B.Sc. IV SEMESTER

Mathematics

PAPER – II

GROUP THEORY, FOURIER SERIES AND DIFFERENTIAL EQUATIONS

UNIT-V

Differential Equation - IV

Syllabus:

Unit – V

Homogeneous linear differential equation of n^{th} order and Equation reducible to the homogeneous linear form, higher order exact differential equations.

-10HRS

Lecture Notes
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5.1. Homogeneous linear differential equation of nth order:

Definition 5.1.1: A differential equation of the form

$$a_0 x^n \frac{d^n y}{d x^n} + a_1 x^{n-1} \frac{d^{n-1} y}{d x^{n-1}} + ...+ a_{n-2} x^2 \frac{d^2 y}{d x^2} + a_{n-1} x \frac{d y}{d x} + a_n y = f(x) \quad (1)$$

where $a_0 , a_{1,}, ... a_{n-1,}, a_n$ are constants is called homogeneous linear differential equation of n^{th} order and is also is called Cauchy-Euler equation.

5.1.2 Reducible to homogeneous linear differential equation of nth order (Cauchy-Euler equation) to linear differential equation of nth order with constant coefficients:

Consider homogeneous linear differential equation of n^{th} order of the form

$$a_0 x^n \frac{d^n y}{d x^n} + a_1 x^{n-1} \frac{d^{n-1} y}{d x^{n-1}} + ...+ a_{n-2} x^2 \frac{d^2 y}{d x^2} + a_{n-1} x \frac{dy}{d x} + a_n y = f(x) \quad (1)$$

where $a_0 , a_{1,}, ... a_{n-1,}, a_n$ are constants.

We transform the Eq. (1) to linear differential equation of n^{th} order with constant coefficients by changing the independent variable x to z i.e.

$$x = e^z \quad \text{or} \quad z = \log x \quad \text{and} \quad \frac{d z}{d x} = \frac{1}{x}$$

Now, $\dfrac{d y}{d x} = \dfrac{d y}{d z}.\dfrac{d z}{d x} = \dfrac{d y}{d z}.\dfrac{1}{x}$

$$\therefore \quad \frac{d y}{d x} = \frac{1}{x}\frac{d y}{d z} \Rightarrow x\frac{d y}{d x} = \frac{d y}{d z}$$

and $\dfrac{d^2 y}{d x^2} = \dfrac{d}{d x}\left(\dfrac{d y}{d x}\right) = \dfrac{d}{d x}\left(\dfrac{1}{x}\dfrac{d y}{d z}\right)$

$$= \frac{1}{x}\frac{d}{d x}\left(\frac{d y}{d z}\right) - \frac{1}{x^2}\frac{d y}{d z}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\left(\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dz} \quad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

On putting $\frac{d}{dz} = D$,

Since $x\frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \therefore x\frac{dy}{dx} = Dy$ (2)

and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = \left(D^2 - D\right)y = D(D-1)y$

$$\therefore x^2\frac{d^2y}{dx^2} = D(D-1)y \quad (3)$$

In general, $x^m\frac{d^my}{dx^m} = D(D-1)..........(D-(m-1))y$ (4)

Substitute Eqs.(2), (3) and (4) in Eq. (1) i.e.

$$a_0D(D-1)......(D-(n-1))y + a_1D(D-1)........(D-(n-2))y + \\ ...... + a_{n-2}D(D-1)y + a_{n-1}Dy + a_ny = f\left(e^z\right)$$

$$\Rightarrow \begin{bmatrix} a_0D(D-1)......(D-(n-1)) + a_1D(D-1)........(D-(n-2)) + \\ ...... + a_{n-2}D(D-1) + a_{n-1}D + a_n \end{bmatrix} y = f\left(e^z\right)$$

$$\Rightarrow F(D)y = f\left(e^z\right) \quad (5)$$

where

$$F(D) = \begin{matrix} a_0D(D-1)......(D-(n-1)) + a_1D(D-1)........(D-(n-2)) + \\ ...... + a_{n-2}D(D-1) + a_{n-1}D + a_n \end{matrix}.$$

Which is the linear differential equation of n^{th} order with constant coefficients in y and z can be solved.

If $y = \psi(z)$ be a solution of Eq. (5), then the solution of Eq. (1) is $y = \psi(\log x) \quad (\because z = \log x)$.

Example: Solve the following

1. $x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = 0$

2. $x^3 D^3 y + 2x^2 D^2 y + 2y = 0$

3. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = x$

4. $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x$

5. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2}{x^3}y = 1$

Solution:

1. The given equation is $x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = 0$ (1)

This is the homogeneous linear differential equation.

Now, $x = e^z$ or $z = \log x$

$\therefore\ x\dfrac{dy}{dx} = Dy$ & $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$, $\left(\dfrac{d}{dz} = D\right)$

Eq. (1) becomes

$$D(D-1)y + 7Dy + 5y = 0$$

$$\Rightarrow (D(D-1) + 7D + 5)y = 0$$

$$\Rightarrow (D^2 - D + 7D + 5)y = 0$$

$$\Rightarrow (D^2 + 6D + 5)y = 0 \quad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $m^2 + 6m + 5 = 0 \Rightarrow m^2 + 5m + m + 5 = 0$

$\Rightarrow m(m+5) + (m+5) = 0 \Rightarrow (m+1)(m+5) = 0$

$\Rightarrow m = -1$ & $m = -5$. The roots are real and distinct.

Therefore, the solution of Eq. (2) is $y = c_1 e^{-z} + c_2 e^{-5z}$ (3)

But $z = \log x$, Eq. (3) becomes

i.e. $y = c_1 e^{-\log x} + c_2 e^{-5\log x} = c_1 e^{\log x^{-1}} + c_2 e^{\log x^{-5}}$

$$= c_1 x^{-1} + c_2 x^{-5}$$

$\therefore\ y = c_1 x^{-1} + c_2 x^{-5}$ is the required solution of Eq. (1).

2. The given equation is $x^3 D^3 y + 2x^2 D^2 y + 2y = 0$ (1)

This is the homogeneous linear differential equation.

Now, $x = e^z$ or $z = \log x$

$\therefore\ xDy = D_1 y,\ x^2 D^2 y = D_1(D_1 - 1)y$ &

$$x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)y \left(\frac{d}{dx} = D \ \& \ \frac{d}{dz} = D_1 \right)$$

Eq. (1) becomes

$$D_1(D_1 - 1)(D_1 - 2)y + 2D_1(D_1 - 1)y + 2y = 0$$

$$\Rightarrow \left(D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 2\right)y = 0$$

$$\Rightarrow \left((D_1^2 - D_1)(D_1 - 2) + 2(D_1^2 - D_1) + 2\right)y = 0$$

$$\Rightarrow \left(D_1^3 - 2D_1^2 - D_1^2 + 2D_1 + 2D_1^2 - 2D_1 + 2\right)y = 0$$

$$\Rightarrow \left(D_1^3 - D_1^2 + 2\right)y = 0 \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $m^3 - m^2 + 2 = 0$ a cubic equation.

Let $m = 1$, $1 - 1 + 2 = 2 \neq 0$ is not a root

& $m = -1$, $-1 - 1 + 2 = 0$ is a root

By Synthetic division,

	1	-1	0	2
-1		-1	2	-2
	1	-2	2	0

The cubic equation can be rewritten as $(m + 1)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -1$ & $m = 1 \pm i$.

One root is real and other root is complex i.e. it occurs in pairs.

Therefore, the solution of Eq. (2) is $y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$ (3)

But $z = \log x$, Eq. (3) becomes

i.e. $y = c_1 e^{-\log x} + e^{\log x} \left(c_2 \cos(\log x) + c_3 \sin(\log x)\right)$

$= c_1 e^{\log x^{-1}} + e^{\log x} \left(c_2 \cos(\log x) + c_3 \sin(\log x)\right)$

$= c_1 x^{-1} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right)$

$\therefore\ y = c_1 x^{-1} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right)$ is the required solution of Eq. (1).

3. The given equation is $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 20y = x$. (1)

This is the homogeneous linear differential equation.

Now, $x = e^z$ or $z = \log x$

$$\therefore\ x\frac{dy}{dx} = Dy \quad \& \quad x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \left(\because \frac{d}{dz} = D\right)$$

Eq. (1) becomes

$$D(D-1)y + 2Dy - 20y = e^z$$

$$\Rightarrow \left(D(D-1) + 2D - 20\right)y = e^z$$

$$\Rightarrow \left(D^2 - D + 2D - 20\right)y = e^z$$

$$\Rightarrow \left(D^2 + D - 20\right)y = e^z \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $m^2 + m - 20 = 0 \Rightarrow m^2 + 5m - 4m - 20 = 0$

$\Rightarrow m(m+5) - 4(m+5) = 0 \Rightarrow (m-4)(m+5) = 0$

$\Rightarrow m = 4 \ \& \ m = -5$, the roots are real and different.

C. F. $= c_1e^{4z} + c_2e^{-5z}$

& P.I. $= \dfrac{1}{D^2 + D - 20}e^z = \dfrac{1}{(1)^2 + (1) - 20}e^z$

$$= \frac{1}{1+1-20}e^z = \frac{1}{-18}e^z = -\frac{1}{18}e^z$$

The solution of Eq. (2) is $y = \text{C.F.} + \text{P.I.}$

$$= c_1e^{4z} + c_2e^{-5z} - \frac{1}{18}e^z \qquad (3)$$

But $z = \log x$, Eq. (3) becomes

i.e. $y = c_1e^{4\log x} + c_2e^{-5\log x} - \dfrac{1}{18}e^{\log x}$

$$= c_1e^{\log x^4} + c_2e^{\log x^{-5}} - \frac{1}{18}e^{\log x}$$

$$= c_1x^4 + c_2x^{-5} - \frac{1}{18}x$$

$\therefore\ y = c_1x^4 + c_2x^{-5} - \dfrac{1}{18}x$ is the required solution of Eq. (1).

4. The given equation is $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x.$ (1)

This is the homogeneous linear differential equation.

Now, $x = e^z$ or $z = \log x$

$$\therefore\ x\frac{dy}{dx} = Dy \quad \& \quad x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \left(\because \frac{d}{dz} = D\right)$$

Eq. (1) becomes

$$D(D-1)y + 5Dy + 4y = ze^z$$

$$\Rightarrow \quad D(D-1)y + 5Dy + 4y = ze^z$$

$$\Rightarrow \quad \left(D(D-1) + 5D + 4\right)y = ze^z$$

$$\Rightarrow \quad \left(D^2 - D + 5D + 4\right)y = ze^z$$

$$\Rightarrow \quad \left(D^2 + 4D + 4\right)y = e^{3z}$$

$$\Rightarrow \quad (D + 2)^2 y = e^{3z} \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $(m + 2)^2 = 0 \Rightarrow m = -2, -2.$ The roots are real and equal.

$$\text{C.F.} = (c_1 + c_2 z)e^{-2z}$$

& $$\text{P.I.} = \frac{1}{(D+2)^2} ze^z = e^z \frac{1}{\left((D+1) + 2\right)^2} z$$

$$= e^z \frac{1}{(D+3)^2} z = e^z \frac{1}{D^2 + 6D + 9} z$$

$$= e^z \frac{1}{9\left(1 + \frac{D^2}{9} + \frac{6}{9}D\right)} z = e^z \frac{1}{9\left[1 + \left(\frac{D^2}{9} + \frac{2}{3}D\right)\right]} z$$

$$= \frac{e^z}{9}\left[1 + \left(\frac{D^2}{9} + \frac{2}{3}D\right)\right]^{-1} z = \frac{e^z}{9}\left[1 - \left(\frac{D^2}{9} + \frac{2}{3}D\right)\right] z$$

$$= \frac{e^z}{9}\left[1 - \frac{D^2}{9} - \frac{2}{3}D\right] z = \frac{e^z}{9}\left[z - \frac{D^2 z}{9} - \frac{2}{3}Dz\right]$$

$$= \frac{e^z}{9}\left[z - 0 - \frac{2}{3}\right] = \frac{e^z}{9}\left[z - \frac{2}{3}\right] = \frac{e^z}{9}\left[\frac{3z - 2}{3}\right]$$

$$= \frac{e^z}{27}(3z - 2)$$

The solution of Eq. (2) is $y = \text{C.F.} + \text{P.I.}$

$$= (c_1 + c_2 z)e^{-2z} + \frac{e^z}{27}(3z - 2) \qquad (3)$$

But $z = \log x$, Eq. (3) becomes

i.e. $$y = \left(c_1 + c_2 (\log x)\right)e^{-2\log x} + \frac{e^{\log x}}{27}\left(3(\log x) - 2\right)$$

$$= (c_1 + c_2 \log x)e^{\log x^{-2}} + \frac{e^{\log x}}{27}(3\log x - 2)$$

$$= (c_1 + c_2 \log x)x^{-2} + \frac{x}{27}(3\log x - 2)$$

$\therefore \quad y = (c_1 + c_2 \log x) x^{-2} + \dfrac{x}{27}(3\log x - 2)$ is the required solution of Eq. (1).

5. The given equation is $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2}{x^3}y = 1.$

Multiply by x^3 i.e. $x^3\dfrac{d^3y}{dx^3} - 4x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} - 2y = x^3$ (1)

This is the homogeneous linear differential equation.

Now, $x = e^z$ or $z = \log x$

$$\therefore x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \&$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \quad \left(\because \frac{d}{dz} = D\right)$$

Eq. (1) becomes

$$D(D-1)(D-2)y - 4D(D-1)y + 5Dy - 2y = (e^z)^3$$

$$\Rightarrow (D(D-1)(D-2) - 4D(D-1) + 5D - 2)y = e^{3z}$$

$$\Rightarrow ((D^2 - D)(D-2) - 4(D^2 - D) + 5D - 2)y = e^{3z}$$

$$\Rightarrow (D^3 - 2D^2 - D^2 + 2D - 4D^2 + 4D + 5D - 2)y = e^{3z}$$

$$\Rightarrow (D^3 - 7D^2 + 11D - 2)y = e^{3z} \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $m^3 - 7m^2 + 11m - 2 = 0$ a cubic equation.

Let $m = 1,\ 1 - 7 + 11 - 2 = 3 \neq 0$ is not a root.

$m = -1,\ -1 - 7 - 11 - 2 = -21 \neq 0$ is not a root.

& $m = 2,\ 8 - 28 + 22 - 2 = 0$ is a root.

By Synthetic division,

	1	- 7	11	- 2
2		2	-10	2
	1	- 5	1	0

The cubic equation can be rewritten as $(m - 2)(m^2 - 5m + 1) = 0$

$$\Rightarrow m = 2 \quad \& \quad m = \frac{5 \pm \sqrt{21}}{2} = \frac{5}{2} \pm \frac{\sqrt{21}}{2}.$$

One root is real and other root is irrational i.e. it occurs in pairs.

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{\left(\frac{5}{2} + \frac{\sqrt{21}}{2}\right)z} + c_3 e^{\left(\frac{5}{2} - \frac{\sqrt{21}}{2}\right)z}$$

$$= c_1 e^{2z} + e^{\frac{5}{2}z}\left(c_2 e^{\frac{\sqrt{21}}{2}z} + c_3 e^{-\frac{\sqrt{21}}{2}z} \right)$$

$$\& \quad \text{P.I.} = \frac{1}{D^3 - 7D^2 + 11D - 2} e^{3z} = \frac{1}{(3)^3 - 7(3)^2 + 11(3) - 2} e^{3z}$$

$$= \frac{1}{27 - 63 + 33 - 2} e^{3z} = \frac{1}{-5} e^{3z} = -\frac{1}{5} e^{3z}$$

The solution of Eq. (2) is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{2z} + e^{\frac{5}{2}z}\left(c_2 e^{\frac{\sqrt{21}}{2}z} + c_3 e^{-\frac{\sqrt{21}}{2}z} \right) - \frac{1}{5} e^{3z} \qquad (3)$$

But $z = \log x$, Eq. (3) becomes

$$\text{i.e.} \quad y = c_1 e^{2\log x} + e^{\frac{5}{2}\log x}\left(c_2 e^{\frac{\sqrt{21}}{2}\log x} + c_3 e^{-\frac{\sqrt{21}}{2}\log x} \right) - \frac{1}{5} e^{3\log x}$$

$$= c_1 e^{\log x^2} + e^{\log x^{5/2}}\left(c_2 e^{\log x^{\sqrt{21}/2}} + c_3 e^{\log x^{-\sqrt{21}/2}} \right) - \frac{1}{5} e^{\log x^3}$$

$$= c_1 x^2 + x^{5/2}\left(c_2 x^{\sqrt{21}/2} + c_3 x^{-\sqrt{21}/2} \right) - \frac{1}{5} x^3$$

$\therefore \quad y = c_1 x^2 + x^{5/2}\left(c_2 x^{\sqrt{21}/2} + c_3 x^{-\sqrt{21}/2} \right) - \frac{1}{5} x^3$ is the required solution of Eq. (1).

Definition 5.1.3 : A differential equation of the form

$$a_0 (a + bx)^n \frac{d^n y}{dx^n} + a_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_{n-2} (a + bx)^2 \frac{d^2 y}{dx^2} + a_{n-1}(a + bx)\frac{dy}{dx} + a_n y = f(x) \qquad (1)$$

where $a_0, a_{1,}, \ldots\ldots a_{n-1,}, a_n$ and a, b are constants is called homogeneous linear differential equation of n^{th} order and is also is called Legendre's form of equation.

5.1.4.Reducible to homogeneous linear differential equation of n^th order (Legendre's form of equation) to linear differential equation of n^th order with constant coefficients:

Consider homogeneous linear differential equation of n^{th} order of the form

$$a_0\left(a+bx\right)^n \frac{d^n y}{dx^n} + a_1\left(a+bx\right)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ...+ a_{n-2}\left(a+bx\right)^2\frac{d^2y}{dx^2}$$
$$+ a_{n-1}\left(a+bx\right)\frac{dy}{dx} + a_n y = f\left(x\right) \tag{1}$$

where a_0 , $a_{1,}$, ... $a_{n-1,}$, a_n and a , b are constants are constants.

We transform the Eq. (1) to linear differential equation of n^{th} order with constant coefficients by changing the independent variable x to z i.e.

$$a + bx = e^z \quad \text{or} \quad z = \log\left(a+bx\right) \quad \text{and} \quad \frac{dz}{dx} = \frac{b}{a+bx}$$

Now, $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}.\left(\frac{b}{a+bx}\right)$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{b}{a+bx}\right)\frac{dy}{dz} \quad \Rightarrow \quad \left(a+bx\right)\frac{dy}{dx} = b\frac{dy}{dz}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\left(\frac{b}{a+bx}\right)\frac{dy}{dz}\right)$

$$= \left(\frac{b}{a+bx}\right)\frac{d}{dx}\left(\frac{dy}{dz}\right) - \left(\frac{b}{\left(a+bx\right)^2}\times b\right)\frac{dy}{dz}$$

$$= \left(\frac{b}{a+bx}\right)\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \left(\frac{b^2}{\left(a+bx\right)^2}\right)\frac{dy}{dz}$$

$$= \left(\frac{b}{a+bx}\right)\frac{d^2y}{dz^2}\left(\frac{b}{a+bx}\right) - \left(\frac{b^2}{\left(a+bx\right)^2}\right)\frac{dy}{dz} \quad \left(\because \frac{dz}{dx} = \frac{b}{a+bx}\right)$$

$$= \left(\frac{b^2}{\left(a+bx\right)^2}\right)\frac{d^2y}{dz^2} - \left(\frac{b^2}{\left(a+bx\right)^2}\right)\frac{dy}{dz} = \left(\frac{b^2}{\left(a+bx\right)^2}\right)\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \left(\frac{b^2}{\left(a+bx\right)^2}\right)\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow \left(a+bx\right)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

On putting $\frac{d}{dz} = D$,

Since $\left(a+bx\right)\frac{dy}{dx} = b\frac{dy}{dz} = bDy \quad \therefore \quad \left(a+bx\right)\frac{dy}{dx} = bDy \quad (2)$

and $\quad (a+bx)^2 \dfrac{d^2y}{dx^2} = b^2\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right) = b^2\left(D^2 y - Dy\right) = b^2\left(D^2 - D\right)y = b^2 D(D-1)y$

$$\therefore \ (a+bx)^2 \frac{d^2y}{dx^2} = b^2 D(D-1)y \qquad (3)$$

In general, $\quad (a+bx)^m \dfrac{d^m y}{dx^m} = b^m D(D-1)..........(D-(m-1))y \qquad (4)$

Substitute Eqs.(2), (3) and (4) in Eq. (1) i.e.

$$a_0 b^n D(D-1)......(D-(n-1))y + a_1 b^{n-1} D(D-1)........(D-(n-2))y +$$

$$...... + a_{n-2} b^2 D(D-1)y + a_{n-1} bDy + a_n y = f\left(\frac{e^z - a}{b}\right)$$

$$\left(\because \ a + bx = e^z \Rightarrow x = \frac{e^z - a}{b}\right)$$

$$\Rightarrow \begin{bmatrix} a_0 b^n D(D-1)...(D-(n-1)) + a_1 b^{n-1} D(D-1)...(D-(n-2)) \\ + ... + a_{n-2} b^2 D(D-1) + a_{n-1} bD + a_n \end{bmatrix} y = f\left(\frac{e^z - a}{b}\right)$$

$$\Rightarrow F(D)y = f\left(\frac{e^z - a}{b}\right) \qquad (5)$$

where

$$F(D) = \begin{matrix} a_0 b^n D(D-1)...(D-(n-1)) + a_1 b^{n-1} D(D-1)...(D-(n-2)) \\ + ... + a_{n-2} b^2 D(D-1) + a_{n-1} bD + a_n \end{matrix}.$$

Which is the linear differential equation of n^{th} order with constant coefficients in y and z can be solved.

If $y = \varphi(z)$ be a solution of Eq. (5), then the solution of Eq. (1) is

$y = \phi(\log(a+bx)) \quad (\because z = \log(a+bx))$.

Example: Solve the following

1. $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$

2. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 0$

3. $(x+a)^2 \dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$

Solution:

1. The given equation is $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0 \quad (1)$

This is the homogeneous linear differential equation.

Now, $5 + 2x = e^z$ or $z = \log(5 + 2x)$

$$\therefore (5+2x)\frac{dy}{dx} = 2Dy \;\&\; (5+2x)^2\frac{d^2y}{dx^2} = 2^2D(D-1)y \quad \left(\frac{d}{dz} = D \;\&\; b = 2\right)$$

$$\Rightarrow (5+2x)\frac{dy}{dx} = 2Dy \;\&\; (5+2x)^2\frac{d^2y}{dx^2} = 4D(D-1)y$$

Eq. (1) becomes

$$4D(D-1)y - 6(2Dy) + 8y = 0$$

$$\Rightarrow \left(4D^2 - 4D - 12D + 8\right)y = 0$$

$$\Rightarrow \left(4D^2 - 16D + 8\right)y = 0 \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $4m^2 - 16m + 8 = 0 \Rightarrow m^2 - 4m + 2 = 0$

$\Rightarrow m = 2 \pm \sqrt{2}$, the roots are irrational.

Therefore, the solution of Eq. (2) is $y = c_1 e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z}$

$$= e^{2z}\left(c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}\right) \qquad (3)$$

But $z = \log(5+2x)$, Eq. (3) becomes

i.e. $y = e^{2\log(5+2x)}\left(c_1 e^{\sqrt{2}\log(5+2x)} + c_2 e^{-\sqrt{2}\log(5+2x)}\right)$

$$= e^{\log(5+2x)^2}\left(c_1 e^{\log(5+2x)^{\sqrt{2}}} + c_2 e^{\log(5+2x)^{-\sqrt{2}}}\right)$$

$$= (5+2x)^2\left(c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}\right)$$

$\therefore y = (5+2x)^2\left(c_1(5+2x)^{\sqrt{2}} + c_2(5+2x)^{-\sqrt{2}}\right)$ is the required solution of Eq. (1).

2. The given equation is $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 0$ (1)

This is the homogeneous linear differential equation.

Now, $3x + 2 = e^z$ or $z = \log(3x+2)$

$$\therefore (3x+2)\frac{dy}{dx} = 3Dy \;\&\; (3x+2)^2\frac{d^2y}{dx^2} = 3^2D(D-1)y \quad \left(\frac{d}{dz} = D \;\&\; b = 3\right)$$

$$\Rightarrow (3x+2)\frac{dy}{dx} = 3Dy \;\&\; (3x+2)^2\frac{d^2y}{dx^2} = 9D(D-1)y$$

Eq. (1) becomes

$$9D(D-1)y + 3(3Dy) - 36y = 0$$

$$\Rightarrow \left(9D^2 - 9D + 9D - 36\right)y = 0$$

$$\Rightarrow \left(9D^2 - 36\right)y = 0 \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $9m^2 - 36 = 0 \Rightarrow m^2 - 4 = 0 \Rightarrow m^2 = 4$

$\Rightarrow m = \pm 2$, the roots are real and different.

Therefore, the solution of Eq. (2) is $y = c_1 e^{2z} + c_2 e^{-2z}$ (3)

But $z = \log(3x + 2)$, Eq. (3) becomes

i.e. $y = c_1 e^{2\log(3x+2)} + c_2 e^{-2\log(3x+2)}$

$= c_1 e^{\log(3x+2)^2} + c_2 e^{\log(3x+2)^{-2}}$

$= c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$

$\therefore y = c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$ is the required solution of Eq. (1).

3. The given equation is $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$ (1)

This is the homogeneous linear differential equation.

Now, $x + a = e^z$ or $z = \log(x+a)$

$$\therefore (x+a)\frac{dy}{dx} = 1Dy \ \& \ (x+a)^2\frac{d^2y}{dx^2} = 1^2 D(D-1)y \quad \left(\frac{d}{dz} = D \ \& \ b = 1\right)$$

$$\Rightarrow (x+a)\frac{dy}{dx} = Dy \ \& \ (x+a)^2\frac{d^2y}{dx^2} = D(D-1)y$$

Eq. (1) becomes,

$$D(D-1)y - 4(Dy) + 6y = e^z - a \quad (\because x + a = e^z)$$

$$\Rightarrow (D(D-1) - 4D + 6y) = e^z - a$$

$$\Rightarrow (D^2 - D - 4D + 6)y = e^z - a$$

$$\Rightarrow (D^2 - 5D + 6)y = e^z - a \qquad (2)$$

Which is linear differential equation with constant coefficients in y and z.

A.E. is $m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2, 3$, the roots are real and different.

C.F. $= c_1 e^{2z} + c_2 e^{3z}$

& P.I. $= \frac{1}{D^2 - 5D + 6}(e^z - a)$

$$= \frac{1}{D^2 - 5D + 6}e^z - \frac{1}{D^2 - 5D + 6}a$$

$$= \frac{1}{D^2 - 5D + 6}e^z - a\frac{1}{D^2 - 5D + 6}e^{0z}$$

$$= \frac{1}{(1)^2 - 5(1) + 6}e^z - a\frac{1}{(0)^2 - 5(0) + 6}e^{0z}$$

$$= \frac{1}{1-5+6}e^{z} - a\frac{1}{0-0+6}$$

$$= \frac{1}{2}e^{z} - \frac{a}{6}$$

The solution of Eq. (2) is $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}e^{z} - \frac{a}{6} \qquad (3)$$

But $z = \log(x+a)$, Eq. (3) becomes

i.e. $y = c_1 e^{2\log(x+a)} + c_2 e^{3\log(x+a)} + \frac{1}{2}e^{\log(x+a)} - \frac{a}{6}$

$$= c_1 e^{\log(x+a)^2} + c_2 e^{\log(x+a)^3} + \frac{1}{2}e^{\log(x+a)} - \frac{a}{6}$$

$$= c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{2}(x+a) - \frac{a}{6}$$

$\therefore\ y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{2}(x+a) - \frac{a}{6}$ is the required solution of Eq. (1).

5.2.Derivation of Condition for Exactness of the Linear Differential Equations:

$$P_0\frac{d^3y}{dx^3} + P_1\frac{d^2y}{dx^2} + P_2\frac{dy}{dx} + P_3 y = f(x) \qquad (1)$$

Where P_0, P_1, P_2, P_3 are functions of x or constants.

If it is an exact differential equation it must have been obtained from an equation of next lower order, simply by differentiation. Since the first term is $P_0\frac{d^3y}{dx^3}$ which can be obtained by differentiation of $P_0\frac{d^2y}{dx^2}$.

Let us assume the solution of the differential equation (1) be

$$P_0\frac{d^2y}{dx^2} + Q_1\frac{dy}{dx} + Q_2 y = \int f(x)\,dx + c \qquad (2)$$

We now find the condition of exactness by using the fact that, differentiating (2) is given by (1).

$$\left[P_0\frac{d^3y}{dx^3}+P_0'\frac{d^2y}{dx^2}\right] + \left[Q_1\frac{d^2y}{dx^2} + Q_1'\frac{dy}{dx}\right] + \left[Q_2\frac{dy}{dx} + Q_2' y\right] = f(x)$$

Rearranging the terms

i.e. $$P_0\frac{d^3y}{dx^3} + \left[P_0' + Q_1\right]\frac{d^2y}{dx^2} + \left[Q_1' + Q_2\right]\frac{dy}{dx} + Q_2' y = f(x) \qquad (3)$$

Comparing (1) and (3) i.e. coefficients for $\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}, y$

$\therefore\ P_0 = P_0$

$P_1 = P_0' + Q_1$

$P_2 = Q_1' + Q_2$

$P_3 = Q_2'$

Since $P_1 = P_0' + Q_1 \quad \Rightarrow \quad Q_1 = P_1 - P_0' = P_1 - P_0'$

$$\therefore\ Q_1 = P_1 - P_0'$$

$P_2 = Q_1' + Q_2 \Rightarrow Q_2 = P_2 - Q_1' = P_2 - \left(P_1 - P_0'\right)' = P_2 - \left(P_1' - P_0''\right)$

$$= P_2 - P_1' + P_0''$$

$$\therefore\ Q_2 = P_2 - P_1' + P_0''$$

$P_3 = Q_2' = \left(P_2 - P_1' + P_0''\right)' = P_2' - P_1'' + P_0'''$

$$\therefore\ P_3 = P_2' - P_1'' + P_0'''$$

$$\Rightarrow\ P_3 - P_2' + P_1'' - P_0''' = 0 \qquad (4)$$

The given equation (1) satisfies the above condition i.e. (4) then the differential equation is said to be exact and its solution i.e. equation (2) becomes

$$P_0\frac{d^2y}{dx^2} + \left(P_1 - P_0'\right)\frac{dy}{dx} + \left(P_2 - P_1' + P_0''\right)y = \int f(x)\,dx + c \qquad (5)$$

This is the solution of equation (1).

Example-1: Show that the equation $\sin x\frac{d^2y}{dx^2} - \cos x\frac{dy}{dx} + 2y\sin x = 0$ is exact.

Solution: Here $P_0 = \sin x,\ P_1 = -\cos x,\ P_2 = 2\sin x.$

Since the equation is of second order, the condition for exactness is

$P_2 - P_1' + P_0'' = 0$.

Now $P_2 - P_1' + P_0'' = 2\sin x - \sin x - \sin x$

$= 2\sin x - 2\sin x = 0$

$\Rightarrow$ the equation is exact.

Example-2: Show that the equation $\left(1-x^2\right)\dfrac{d^2 y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$ is exact and solve.

Solution: Here $P_0 = 1 + x^2$, $P_1 = 3x$, $P_2 = 1$.

Since the equation is of second order, the condition for exactness is

$P_2 - P_1' + P_0'' = 0$.

Now $P_2 - P_1' + P_0'' = -1+3-2 = 3-3 = 0$

$\Rightarrow$ the equation is exact.

Therefore the solution is $P_0\dfrac{dy}{dx} + \left(P_1 - P_0'\right)y = \int 0\,dx + c_1$

$$\Rightarrow \left(1-x^2\right)\frac{dy}{dx} + \left(-3x+2x\right)y = \int 0\,dx + c_1$$

$$\Rightarrow \left(1-x^2\right)\frac{dy}{dx} - xy = c_1$$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{c_1}{1-x^2}$$

Which is linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$ with

$P = -\dfrac{x}{1-x^2}$, $Q = \dfrac{c_1}{1-x^2}$.

Now I.F. $= e^{\int P\,dx} = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\log\left(1-x^2\right)} = e^{\log\left(1-x^2\right)^{\frac{1}{2}}} = \left(1-x^2\right)^{\frac{1}{2}}$

and the solution is

$y(I.F.) = \int (I.F.)\,Q\,dx + c_2$

$$\Rightarrow y\left(\left(1-x^2\right)^{\frac{1}{2}}\right) = \int \left(\left(1-x^2\right)^{\frac{1}{2}}\right)\left(\frac{c_1}{1-x^2}\right)dx + c_2$$

or $y\sqrt{1-x^2} = \int \sqrt{1-x^2}\left(\dfrac{c_1}{1-x^2}\right) dx + c_2$

$$\Rightarrow y\sqrt{1-x^2} = \int \frac{c_1}{\sqrt{1-x^2}}\, dx + c_2$$

$$\Rightarrow y\sqrt{1-x^2} = c_1 \int \frac{1}{\sqrt{1-x^2}}\, dx + c_2$$

$$\Rightarrow y\sqrt{1-x^2} = c_1 \sin^{-1} x + c_2$$

This is the required solution.

Example-3:Solve $x^2 \dfrac{d^2 y}{dx^2} + x\dfrac{dy}{dx} - y = x^3$.

Solution: Here $P_0 = x^2,\; P_1 = x,\; P_2 = -1$.

Since the equation is of second order, the condition for exactness is

$P_2 - P_1' + P_0'' = 0$.

Now $P_2 - P_1' + P_0'' = -1 - 1 + 2 = 0 \Rightarrow$ the equation is exact.

Therefore the solution is $P_0 \dfrac{dy}{dx} + \left(P_1 - P_0'\right) y = \int x^3\, dx + c_1$

$$\Rightarrow x^2 \frac{dy}{dx} + (x - 2x)\, y = \frac{x^4}{4} + c_1$$

$$\Rightarrow x^2 \frac{dy}{dx} - xy = \frac{x^4}{4} + c_1$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} y = \frac{x^2}{4} + \frac{c_1}{x^2}.$$

Which is linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$ with

$P = -\dfrac{1}{x},\; Q = \dfrac{x^2}{4} + \dfrac{c_1}{x^2}$.

Now I.F. $= e^{\int P\, dx} = e^{\int -\frac{1}{x}\, dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \dfrac{1}{x}$

and the solution is

$$y(I.F.) = \int (I.F.)\, Q\, dx + c_2$$

$$\Rightarrow y\left(\frac{1}{x}\right)=\int\left(\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{c_1}{x^2}\right)dx+c_2=\int\left(\frac{x}{4}+\frac{c_1}{x^3}\right)dx+c_2$$

$$\Rightarrow y\left(\frac{1}{x}\right)=\frac{x^2}{8}-\frac{c_1}{2x^2}+c_2$$

$$\Rightarrow y=\frac{x^3}{8}-\frac{c_1}{2x}+c_2x.$$

This is the required solution.

Example-4:Solve $\left(1+x+x^2\right)\frac{d^3y}{dx^3}+\left(3+6x\right)\frac{d^2y}{dx^2}+6\frac{dy}{dx}=0.$

Solution: Here $P_0=1+x+x^2,\ P_1=3+6x,\ P_2=6,\ P_3=0$.

Since the equation is of third order, the condition for exactness is

$$P_3-P_2'+P_1''-P_0'''=0.$$

Now $P_3-P_2'+P_1''-P_0'''=0-0+0+0=0\Rightarrow$ the equation is exact.

Therefore the solution is

$$P_0\frac{d^2y}{dx^2}+\left(P_1-P_0'\right)\frac{dy}{dx}+\left(P_2-P_1'+P_0''\right)y=c_1$$

$$\Rightarrow\left(1+x+x^2\right)\frac{d^2y}{dx^2}+\left(3+6x-\left(1+2x\right)\right)\frac{dy}{dx}+\left(6-6+2\right)y=c_1$$

$$\Rightarrow\left(1+x+x^2\right)\frac{d^2y}{dx^2}+\left(2+4x\right)\frac{dy}{dx}+2y=c_1$$ which is second order.

Again, here $P_0=1+x+x^2,\ P_1=2+4x,\ P_2=2$ and the condition for exactness is $P_2-P_1'+P_0''=0$.

Now, $P_2-P_1'+P_0''=2-4+2=0\Rightarrow$ the equation is exact.

Therefore the solution is

$$P_0\frac{dy}{dx}+\left(P_1-P_0'\right)y=\int c_1\,dx+c_2$$

$$\Rightarrow\left(1+x+x^2\right)\frac{dy}{dx}+\left(2+4x-\left(1+2x\right)\right)y=c_1x+c_2$$

$$\Rightarrow\left(1+x+x^2\right)\frac{dy}{dx}+\left(1+2x\right)y=c_1x+c_2$$

$$\Rightarrow \frac{d}{dx}\left[\left(1 + x + x^2\right)y\right] = c_1 x + c_2$$

$$\Rightarrow \left(\left(1 + x + x^2\right)\right)y = \int\left(c_1 x + c_2\right)dx + c_3$$

$$\Rightarrow \left(1 + x + x^2\right)y = c_1 \frac{x^2}{2} + c_2 x + c_3$$

This is the required solution.

5.2.1.Extension for Condition of Exactness

Condition for Exactness of the nth order Linear Differential Equations:

Consider the n^{th} order linear differential equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + P_n y = f(x) \qquad (1)$$

Where P_0 , $P_1, P_2, ...P_n$ are functions of x .

5.2.2.Derivation of Condition for Exactness:

If it is an exact differential equation it must have been obtained from an equation of next lower order, simply by differentiation. Since the first term is $P_0 \frac{d^n y}{dx^n}$ which can be obtained by differentiation of $P_0 \frac{d^{n-1}y}{dx^{n-1}}$.

Let us assume the solution of the differential equation (1) be

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + Q_1 \frac{d^{n-2}y}{dx^{n-2}} + Q_2 \frac{d^{n-3}y}{dx^{n-3}} + \dots\dots\dots + Q_{n-1} y = \int f(x)\,dx + c \qquad (2)$$

We now find the condition of exactness by using the fact that, differentiating (2) is given by (1).

$$\left[P_0\frac{d^n y}{dx^n} + P_0'\frac{d^{n-1}y}{dx^{n-1}}\right] + \left[Q_1\frac{d^{n-1}y}{dx^{n-1}} + Q_1'\frac{d^{n-2}y}{dx^{n-2}}\right] +$$

Rearr

$$\left[Q_2\frac{d^{n-2}y}{dx2} + Q_2'\frac{d^{n-2}y}{dx2}\right] + \ldots\ldots\ldots + \left[Q_{n-1}\frac{dy}{dx} + Q_{n-1}'y\right] = f(x)$$

anging the terms

i.e. $P_0\frac{d^n y}{dx^n} + \left[P_0' + Q_1\right]\frac{d^{n-1}y}{dx^{n-1}} + \left[Q_1' + Q_2\right]\frac{d^{n-2}y}{dx^{n-2}} +$

$$\left[Q_2' + Q_3\right]\frac{d^{n-3}y}{dx^{n-3}} + \ldots\ldots\ldots + \left[Q_{n-2}' + Q_{n-1}\right]\frac{dy}{dx} + Q_{n-1}'y =$$

$$\left[Q_2' + Q_3\right]\frac{d^{n-3}y}{dx^{n-3}} + \ldots. + \left[Q_{n-2}' + Q_{n-1}\right]\frac{dy}{dx} + Q_{n-1}'y = f(x) \qquad (3)$$

Comparing (1) and (3) i.e. coefficients for $\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots\ldots y$

$\therefore\ P_0 = P_0$

$P_1 = P_0' + Q_1$

$P_2 = Q_1' + Q_2$

……….

……….

$P_{n-1} = Q_{n-2}' + Q_{n-1}$ and $P_n = Q_{n-1}'$

Since $P_1 = P_0' + Q_1 \Rightarrow Q_1 = P_1 - P_0' = P_1 - (-1)^{1-1}P_0'$

$\therefore\ Q_1 = P_1 - (-1)^{1-1}P_0'$

$P_2 = Q_1' + Q_2 \Rightarrow Q_2 = P_2 - Q_1' =$

$$P_2 - \left(P_1 - P_0'\right)' = P_2 - \left(P_1' - P_0''\right) = P_2 - P_1' + P_0''$$

$\therefore\ Q_2 = P_2 - P_1' - (-1)^{2-1}P_0''$

Similarly $Q_3 = P_3 - P_2' + P_1'' - (-1)^{3-1}P_0'''$ ……….

$$Q_{n-1} = P_{n-1} - P'_{n-2} + P''_{n-3} - \dots\dots\dots - (-1)^{(n-1)-1} P_0^{n-1}$$

But

$$P_n = Q'_{n-1} = \left(P_{n-1} - P'_{n-2} + P''_{n-3} - \dots\dots\dots\dots\dots - (-1)^{(n-1)-1} P_0^{n-1} \right)'$$

$$= P'_{n-1} - P''_{n-2} + P'''_{n-3} \dots\dots\dots\dots\dots + (-1)^{n-1} P_0^{n}$$

$$\therefore P_n = P'_{n-1} - P''_{n-2} + P'''_{n-3} \dots\dots\dots\dots\dots + (-1)^{n-1} P_0^{n}$$

$$\Rightarrow P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} \dots\dots\dots\dots\dots - (-1)^{n-1} P_0^{n} = 0 \qquad (4)$$

The given equation (1) satisfies the above condition i.e. (4) then the differential equation is said to be exact and its solution i.e. equation (2) becomes

$$P_0 \frac{d^{n-1}y}{d\,x^{n-1}} + \left(P_1 - P'_0\right) \frac{d^{n-2}y}{d\,x^{n-2}} + \left(P_2 - P'_1 + P''_0\right) \frac{d^{n-3}y}{d\,x^{n-3}} .+ \dots\dots$$
$$+ \left(P_{n-1} - P'_{n-2} + P''_{n-3} - \dots\dots\dots\dots\dots - (-1)^{(n-1)-1} P_0^{n-1}\right) y = \int f(x)\, d\,x + c \qquad (5)$$

This is the solution of equation (1).